Exam Lie Groups in Physics

Date	November 8, 2017
Room	BB 5161.0165
Time	9:00 - 12:00
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- The weights of the **four** problems are given below
- Answers may be given in Dutch
- Illegible handwriting will be graded as incorrect
- Good luck!

Weighting

1a)	6	2a)	10	3a)	6	4a)	6
1b)	6	2b)	10	3b)	9	4b)	9
1c)	6	2c)	10			4c)	6
1d)	6						

$$\text{Result } = \frac{\sum \text{points}}{10} + 1$$

Problem 1
Consider the group $SU(2)$ of all unitary 2×2 matrices with determinant equal to 1, and the group $U(1)$ of all complex phases.

(a) Derive a parametrization of all the elements of the group $SU(2)$, use it to determine the parameter space of $SU(2)$, and explain whether the group is compact and/or connected or neither.

(b) Give a parametrization of a $U(1)$ subgroup of $SU(2)$ in the defining representation of $SU(2)$. Explain whether this is an irreducible representation of $U(1)$ or not.

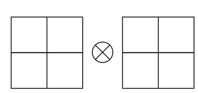
(c) Show whether the coset space $SU(2)/U(1)$ is a factor group or not.

(d) Show that the center of $SU(2)$ is isomorphic to Z_2 (you are allowed to use Schur's lemma, assuming that the defining rep is irreducible) and discuss whether $SU(2)/\mathsf{Z}_2$ is a factor group or not.

Problem 2

Consider the Lie algebra $su(n)$ of the Lie group $SU(n)$ of unitary $n \times n$ matrices with determinant equal to 1.

(a) Decompose the following direct product of irreps of the Lie algebra $su(n)$



into a direct sum of irreps of $su(n)$, in other words, determine its Clebsch-Gordan series. Suggestion: first write down the allowed "words" consisting of two a's and two b's.

(b) Write down the dimensions of the irreps appearing in the obtained decomposition for $su(3)$ and $su(4)$. Indicate the complex conjugate and inequivalent irreps whenever appropriate.

(c) Consider for $su(3)$ the complex conjugate of the above direct product in terms of Young tableaux, decompose it into irreps and compare the answer to the one obtained in part (b).

Problem 3
Consider the four-dimensional representation of the generators of the Lorentz group:

$$(M^{\mu\nu})^{\alpha}{}_{\beta} = i(g^{\mu\alpha}g^{\nu}{}_{\beta} - g^{\nu\alpha}g^{\mu}{}_{\beta})$$

(a) Write down the matrices for the following two cases: $\mu = 0, \nu = 1$ and $\mu = 2, \nu = 3$.

(b) Derive an expression for $\exp(-i\chi M^{01})$ in terms of hyperbolic cosines and sines, using the expression for M^{01} obtained in part (a). Conclude which Lorentz transformation it corresponds to. Recall that $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.

Problem 4
Consider the Poincaré group that consists of all transformations that leave distances between four-vectors invariant.

(a) Describe the connected components of the Poincaré group.

Consider the Pauli-Lubanski vector $W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$, where P_ν is the four-momentum and $M_{\rho\sigma}$ denotes the generators of the Lorentz group.

(b) Give expressions for W^0 and W^1 in terms of the generators P^0, P^i, $J^k = \frac{1}{2}\epsilon_{klm}M^{lm}$ and $K^i = M^{0i}$, using $\epsilon^{0123} = 1$. The indices i, j, k, l, m can take values $1, 2, 3$. Show that W^0 is related to the helicity and write down the obtained expressions in the rest frame of a massive particle.

(c) Demonstrate that $W_\mu P^\mu = 0$.